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Hopf $\star$-deformation of any Lie group

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Abstract

We use the Neroslavsky–Vlassov (1981) method to find a star product $\star_h$ on a class of exact Poisson-Lie groups such that $(C^\infty(G)[[h]], \star_h, \Delta)$ is a Hopf algebra. We show that we can find such a nontrivial star product on every Lie group.

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1. The problem

Let G be a Lie group and $(C^\infty(G), \cdot, \Delta)$ the corresponding Hopf algebra (where $\cdot$ is the usual product of functions on G and $\Delta : C^\infty(G) \to C^\infty(G)\hat{\otimes}C^\infty(G) \equiv C^\infty(G\times G)$ is the usual coproduct with $(\Delta f)(x, y) := f(xy)$). In this note, we will consider the deformations $(C^\infty(G)[[h]], \star_h, \Delta_h)$ of this Hopf algebra such that the coproduct Δ_h, the unit and the counit are the $\mathbb{C}[[h]]$-linear extension of their classical equivalent (we shall write Δ for Δ_h) and such that $\star_h$ is a differential star product.

In this class of deformation Drinfeld has proved the existence of deformation for every exact Poisson-Lie groups $(G, \{\ \}_r)$, where $[r, r] = 0$, and for every simple Poisson-Lie groups.

Some people consider the deformation $(C^\infty(G)[[h]], \star_h, \Delta_h)$ of $(G, \cdot, \Delta)$ where the coproduct Δ_h is not the $\mathbb{C}[[h]]$-linear extension of this classical equivalent Δ. In this class of deformation Eingof and Kazhdan have proved the existence of deformation for every Poisson-Lie group $(G, \{\ \})$.

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Definition. Let $(C^\infty(G)[[h]], \star_h, \Delta)$ be a deformation of the Hopf algebra $(C^\infty(G), \cdot, \Delta)$. Define the following bracket:

$$\{a, b\} := \lim_{h \to 0} \frac{1}{h}(a \star_h b - b \star_h a), \quad \text{where } a, b \in C^\infty(G).$$

Since $\star_h$ is associative and compatible with the coproduct Δ, with this bracket $(G, \{\ \})$ is a Poisson-Lie group. We shall call $(G, \{\ \})$ the *classical limit* of $(C^\infty(G), \star_h, \Delta)$ and $(C^\infty(G), \star_h, \Delta)$ a *Hopf $\star$ deformation* of $(G, \{\ \})$.

Question. Let $(G, \{\ \})$ be a Poisson-Lie group. Does there exist a Hopf $\star$ deformation $(C^\infty(G), \star_h, \Delta)$ of $(G, \{\ \})$?

Remark. (Bonneau et al. [2], Gerstenhaber and Schack [5]). If $(C^\infty(G), \star_h, \Delta)$, is a deformation of the bialgebra $(C^\infty(G), \cdot, \Delta)$ thus it is a Hopf algebra and then a deformation of the Hopf algebra $(C^\infty(G), \cdot, \Delta)$.

Let $(G, \{\ \})$ be a Poisson-Lie group; we are thus looking for a differential star product $\star_h$ on $C^\infty(G)$ such that:

- it is associative:

$$a \star_h (b \star_h c) = (a \star_h b) \star_h c \quad \forall a, b, c \in C^\infty(G);$$

- it admits 1 as unit:

$$a \star_h 1 = 1 \star_h a = a \quad \forall a \in C^\infty(G);$$

- it is compatible with the coproduct Δ and the counit ϵ (where $\epsilon : C^\infty(G) \to \mathbb{R}$ or $\mathbb{C}$ is the usual counit with $\epsilon(f) := f(e)$, e is the neutral element of G):

$$\Delta(a \star_h b) = \Delta(a) \star\star_h \Delta(b), \quad \epsilon(a \star_h b) = \epsilon(a)\epsilon(b) \qquad \forall a, b \in C^\infty(G),$$

  where $\star\star_h$ is the natural extension of $\star_h$ on $\hat{\otimes}^2 C^\infty(G)$ $(a_1 \otimes a_2 \star\star_h b_1 \otimes b_2 := a_1 \star_h b_1 \otimes a_2 \star_h b_2)$;
- its associated Poisson bracket is the Poisson bracket of the Poisson-Lie group:

$$\lim_{h \to 0} \frac{1}{h}(a \star_h b - b \star_h a) = \{a, b\} \quad \forall a, b \in C^\infty(G).$$

The answer to the question of quantization of $(G, \{\ \})$ is not known in general; we refer to [1] for a precise description of the state of the question. We shall study here the exact case – which we recall in Section 2 – translating its quantization into a cohomological problem in Section 3. We solve this problem for some examples; this leads in particular to a nontrivial Hopf $\star$ deformation on any Lie group (for a particular Poisson bracket).

2. The exact case

2.1. Definition

Notation. Let G be a Lie group; we shall denote by $\mathcal{G}$ its Lie algebra, by $\mathcal{UG}$ its enveloping algebra and by $\mathcal{D}$ the space of differential operators on G. We call λ (resp. ρ) the only homomorphism (resp. antihomomorphism) from $\mathcal{UG}$ to $\mathcal{D}$ which maps X on the corresponding left (resp. right) invariant vector field on G.

Definition. A Poisson-Lie group $(G, \{\ \})$ is said to be *exact* if there exists an element $r \in \wedge^2\mathcal{G}$ such that

$$\{a, b\} = m \circ (\otimes^2\lambda(r) - \otimes^2\rho(r))(a \otimes b) \quad \forall a, b \in C^\infty(G),$$

where m is the usual multiplication of functions. In that case, we shall denote $\{\ \} =: \{\ \}_r$.

Remark. With such a bracket $(G, \{\ \}_r)$ is a Poisson-Lie group if and only if $[r, r] \in (\wedge^3\mathcal{G})^{\text{inv}}$ where $[\ ]$ is the Schouten bracket and $(\wedge^3\mathcal{G})^{\text{inv}}$ is the subspace of all the elements of $(\wedge^3\mathcal{G})$ invariant under the adjoint action.

2.2. Takhtajan's Hopf $\star$ deformation

Notation. The following way to describe the elements of $\otimes^k\mathcal{UG}$ will be useful. Let $\mathcal{G}$ be a Lie algebra and $\{X_1, \ldots, X_n\}$ be a basis of $\mathcal{G}$; we shall write x_i for X_i (resp. $X_i \otimes 1$, $X_i \otimes 1 \otimes 1$, etc.) if $k = 1$ (resp. 2, 3, etc.). We shall write y_i for $1 \otimes X_i$ (resp. $1 \otimes X_i \otimes 1$, etc.) if $k = 2$ (resp. 3, etc.). Hence an element of $\otimes^2\mathcal{UG}$ will be written as a polynomial in the noncommuting variables $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$.

Takhtajan considers a star product of the form:

$$a \star_h b := m \circ \otimes^2\rho(F^{-1}(x, y)) \circ \otimes^2\lambda(F(x, y))(a \otimes b) \quad \forall a, b \in C^\infty(G),$$

where $F(x, y) = 1 \otimes 1 + hF_1(x, y) + h^2F_2(x, y) + \cdots \in \otimes^2\mathcal{UG}[[h]]$.

Remarks. *(about the star-product).*
1. The interest to consider such a star product is that it is compatible with the coproduct Δ and the counit ϵ.
2. Such a star product will be associative if [7] and only if:

$$F(x + y, z)F(x, y)F^{-1}(y, z)F^{-1}(x, y + z) \in (\otimes^3\mathcal{UG})^{\text{inv}}[[h]],$$

   where $(\otimes^3\mathcal{UG})^{\text{inv}}$ is the subset of the elements of $\otimes^3\mathcal{UG}$ invariant under the adjoint action.
3. The function 1 will be a unit for this star product if $F_i \in \otimes^2(\mathcal{UG})_0$, for all $i \geq 1$, where $(\mathcal{UG})_0 := \rangle X_1 \cdots X_k \in \mathcal{UG} \mid k \geq 1\langle \subset \mathcal{UG}$.

4. The associated Poisson bracket will be always exact:

$$\lim_{h\to 0}\frac{1}{h}(a\star_h b - b\star_h a)$$
$$= m\circ(\otimes^2\lambda(F_1(x,y)-F_1(y,x)) - \otimes^2\rho(F_1(x,y)-F_1(y,x)))(a\otimes b).$$

   Hence this method provides Hopf $\star$ deformation only for exact Poisson-Lie groups.
5. If $H^1(G,\otimes^2\mathcal{U}\mathcal{G},\delta) = 0$, in particular if G is semi-simple and simply connected or if G is compact, then for any Hopf $\star$ deformation $(G,\star_h,\Delta)$ of $(G,\{\ \})$ there exists $F\in\otimes^2\mathcal{U}\mathcal{G}[[h]]$ such that

$$a\star_h b := m\circ\otimes^2\rho(F^{-1}(x,y))\circ\otimes^2\lambda(F(x,y))(a\otimes b)\quad \forall a,b\in C^\infty(G).$$

Notation. Let V be a vector space and $u=\sum_{i\geq 0}h^i u_i$, $v=\sum_{i\geq 0}h^i v_i$ be two elements of $V[[h]]$. We shall denote by $(u)_n$ the truncated sum $\sum_{i=0}^n h^i u_i$, by $[u]_n$ the nth order u_n and we shall write $u\overset{n}{=}v$ for $(u)_n=(v)_n$.

To prove the above remarks, we proceed as follows. Let $\star_h$ be a differential star product

$$a\star_h b := \sum_{n\,\alpha\,\beta} h^n C_{n\,\alpha\,\beta}(\lambda(x_\alpha)a)(\lambda(x_\beta)b),$$

where $C_{n\alpha\beta}\in C^\infty(G)$ for all $n\in\mathbb{N}$, all $\alpha := (\alpha^1,\ldots,\alpha^k)$, $\beta := (\beta^1,\ldots,\beta^l)$, α^i, $\beta^j\in\mathbb{N}$, and $x_\alpha := x_{\alpha^1}\cdots x_{\alpha^k}$ and $y_\beta := y_{\beta^1}\cdots y_{\beta^l}$. First, we shall write the compatibility condition and find the corresponding condition on C. The induced star product $\star\star_h$ on $C^\infty(G\times G)$ is:

$$(A\star\star_h B)(x,y) = \sum_{n_1 n_2\alpha_1\alpha_2\beta_1\beta_2} h^{n_1+n_2} C_{n_1\alpha_1\beta_1}(x)C_{n_2\alpha_2\beta_2}(y)$$
$$\times(\hat{\otimes}^2\lambda(x_{\alpha_1}y_{\alpha_2})A)(x,y)(\hat{\otimes}^2\lambda(x_{\beta_1}y_{\beta_2})B)(x,y).$$

Since we have

$$(\hat{\otimes}^2\lambda(x_{\alpha_1}y_{\alpha_2})\Delta a)(x,y) = (\lambda((Ad\,y^{-1}x_{\alpha_1})x_{\alpha_2})a)(xy),$$

the right-hand side of the compatibility condition $\Delta(a\star_h b)=\Delta(a)\star\star_h\Delta(b)$ is

$$\sum_{n_1 n_2\alpha_1\alpha_2\beta_1\beta_2} h^{n_1+n_2}C_{n_1\alpha_1\beta_1}(x)$$
$$\times C_{n_2\alpha_2\beta_2}(y)(\lambda((Ad\,y^{-1}x_{\alpha_1})x_{\alpha_2})a)(xy)(\lambda((Ad\,y^{-1}x_{\beta_1})x_{\beta_2})b)(xy).$$

The compatibility condition is equivalent to

$$C(xy) = (Ad\,y^{-1}C(x))\cdot C(y), \tag{1}$$

where $C(x) := \sum_{n\alpha\beta}h^n C_{n\alpha\beta}x_\alpha y_\beta$ and $\cdot$ is the component by component product on $\otimes^2\mathcal{U}\mathcal{G}$.

If $\star_h$ is a Takhtajan star product then $C(x) := (Ad\,x^{-1}F^{-1})\cdot F$ and since this C satisfies condition (1) we have proved the first part of the remark.

Now we shall prove that if $H^1(G, \otimes^2\mathcal{UG}, \delta) = 0$, the only C which satisfies condition (1) is $C(x) := (Ad\, x^{-1}F^{-1}) \cdot F$, where $F \in \otimes^2\mathcal{UG}[[h]]$ and $(F)_0 = 1 \otimes 1$. Indeed:

$$\begin{aligned}
&[(Ad\, y^{-1}C(x)) \cdot C(y) - C(xy)]_n \\
&\quad = C_n(y) - C_n(xy) + Ad\, y^{-1}C_n(x) + bc(C)(x, y) \\
&\quad = (\delta C_n)(x, y) + bc(C)(x, y),
\end{aligned} \tag{2}$$

where $bc(C)$ only depends on C up to order $n - 1$ and

$$[(Ad\, x^{-1}F^{-1}) \cdot F]_n = F_n - Ad\, x^{-1}F_n + bf(F) = (\delta F_n)(x) + bf(F),$$

where $bf(F)$ only depends on F up to order $n - 1$.

At the first order Eq. (1) gives $(\delta C_1)(x, y) = 0$. On the other hand, $[(Ad\, x^{-1}F^{-1}) \cdot F]_1 = (\delta F_1)(x)$. Hence, if $H^1(G, \otimes^2\mathcal{UG}, \delta) = 0$, for each C which satisfies (1), there exists (F_1) defined at order 1 such that $C(x) \stackrel{1}{=} (Ad\, x^{-1}(F^{-1})_1 \cdot (F)_1$. Assume it is true at the nth order; i.e. if C satisfies (1), there exists $(F)_n$ defined at the order n such that

$$C(x) \stackrel{n}{=} (Ad\, x^{-1}(F^{-1})_n \cdot (F)_n$$

then, since $C'(x) \stackrel{n}{:=} (Ad\, x^{-1}(F^{-1})_n \cdot (F)_n$ satisfies Eq. (1) one has for Eq. (2)

$$\delta(C_{n+1}(x) - ((Ad\, x^{-1}(F^{-1})_n \cdot (F)_n)_{n+1}) = 0.$$

Hence, if $H^1(G, \otimes^2\mathcal{UG}, \delta) = 0$,

$$\begin{aligned}
C_{n+1}(x) &= ((Ad\, x^{-1}(F^{-1})_n) \cdot (F)_n)_{n+1} + \delta F_{n+1}, \\
C(x) &\stackrel{n+1}{=} (Ad\, x^{-1}(F^{-1})_{n+1} \cdot (F)_{n+1}
\end{aligned}$$

with $(F)_{n+1} = (F)_n + h^n F_{n+1}$.

Then for each C which satisfies (1), there exists F such that $C(x) = (Ad\, x^{-1}F^{-1}) \cdot F$. We have proved the fifth condition of the remark.

2.3. *The pentagonal equation*

Let $(G, \{\ \}_r)$ be an exact Poisson-Lie group; to build a Hopf $\star$ deformation as described above, we are looking for an element $\alpha = 1 \otimes 1 \otimes 1 + \sum_{i \geq 1} h^i \alpha_i$, where $\alpha_i \in (\otimes^3(\mathcal{UG})_0)^{\mathrm{inv}}$, and an element $F = 1 \otimes 1 + \sum_{i \geq 1} h^i F_i$, where $F_i \in \otimes^2(\mathcal{UG})_0$, such that

$$\begin{aligned}
&F_1(x, y) - F_1(y, x) = r, \\
&A(F, \alpha) := \alpha(x, y, z)F(x, y + z)F(y, z) - F(x + y, z)F(x, y) = 0.
\end{aligned}$$

We shall now study order by order and from a cohomological point of view, the associativity equation $A(F, \alpha) = 0$.

From now on, we assume that α is fixed and we shall see how the cohomological obstruction for the existence of an F which satisfies the associativity equation $A(F, \alpha) = 0$

implies condition on α: the pentagonal equation $P(\alpha) = 0$ where $P(\alpha)$ is defined below (see Eq. (3)).

The nth order of the associativity equation reads:

$$[A(F,\alpha)]_n = dF_n + \alpha_n + a_n(F,\alpha),$$

where d is the Hochschild coboundary operator, and $a_n(F,\alpha)$ depends only on F and α, up to order $n-1$ ($a_n(F,\alpha) = a_n((F)_{n-1},(\alpha)_{n-1})$).

Suppose that $F = 1\otimes 1 + \sum_{i=1}^{n-1} h^i F_i$ is a solution of the associativity equation, up to order $n-1$ ($A(F,\alpha) \overset{n-1}{=} 0$). Then a necessary condition for the existence of a prolongation of this solution at order n is $d(\alpha_n + a_n(F,\alpha)) = 0$.

Proposition 1. *Let* $\alpha = 1\otimes 1\otimes 1 + \sum_{i\geq 1} h^i\alpha_i(x,y,z)$ *and* $F = 1\otimes 1 + \sum_{i=1}^{n-1} h^i F_i(x,y)$ *such that* $A(F,\alpha) \overset{n-1}{=} 0$. *Then* $d(\alpha_n + a_n(F,\alpha)) = 0$ *if and only if*

$$P(\alpha) := \alpha(x,y,z)\alpha(x,y+z,t)\alpha(y,z,t) - \alpha(x+y,z,t)\alpha(x,y,z+t) \overset{n}{=} 0. \quad (3)$$

Definition. Given $F = 1\otimes 1 + \sum_{i\geq 1} h^i F_i$, let us define the function $f(F)$ by

$$f(F) := F(x+y,z)F(x,y)F^{-1}(y,z)F^{-1}(x,y+z). \quad (4)$$

Lemma 1. *Let* $F = 1\otimes 1 + \sum_{i=1}^{n-1} h^i F_i$ *be such that* $A(F,\alpha) \overset{n-1}{=} 0$. *Then*

$$[f(F)]_n = -a_n(F,\alpha).$$

Proof. Since $A(F,\alpha) \overset{n-1}{=} 0$ we have

$$(F(x+y,z)F(x,y))_{n-1} = ((\alpha)_{n-1}F(x,y+z)F(y,z))_{n-1},$$

then

$$\begin{aligned}(F(x+y,z)F(x,y))_n = {}& ((\alpha)_{n-1}F(x,y+z)F(y,z))_n \\ & - h^n[(\alpha)_{n-1}F(x,y+z)F(y,z) - F(x+y,z)F(x,y)]_n.\end{aligned}$$

Further

$$\begin{aligned} f(F) &\overset{n}{=} ((\alpha)_{n-1}F(x,y+z)F(y,z))_n F^{-1}(y,z)F^{-1}(x,y+z) \\ &\quad - h^n[(\alpha)_{n-1}F(x,y+z)F(y,z) - F(x+y,z)F(x,y)]_n \\ &\overset{n}{=} (\alpha)_{n-1} - h^n[(\alpha)_{n-1}F(x,y+z)F(y,z) - F(x+y,z)F(x,y)]_n. \end{aligned}$$

□

Lemma 2. *Assume that* $F = 1\otimes 1 + \sum_{i\geq 1}^{n-1} h^i F_i$ *satisfies* $[f(F)(x,y,z), F(x+y+z,t)] \overset{n}{=} 0$ *and* $[f(F)(y,z,t), F(x,y+z+t)] \overset{n}{=} 0$. *Then* $P(f(F)) \overset{n}{=} 0$.

Proof.

$$\begin{aligned}
f(F) &\stackrel{n}{=} f(F)(x,y,z)F(x+y+z,t)F(x,y+z)\\
&\quad\times F^{-1}(y+z,t)F^{-1}(x,y+z+t)f(F)(y,z,t)\\
&\quad - f(F)(x+y,z,t)f(F)(x,y,z+t)\\
&\stackrel{n}{=} F(x+y+z,t)f(F)(x,y,z)F(x,y+z)\\
&\quad\times F^{-1}(y+z,t)f(F)(y,z,t)F^{-1}(x,y+z+t)\\
&\quad - f(F)(x+y,z,t)f(F)(x,y,z+t)\\
&\stackrel{n}{=} F(x+y+z,t)F(x+y,z)F(x,y)F^{-1}(y,z)F^{-1}(x,y+z)\\
&\quad\times F(x,y+z)F^{-1}(y+z,t)F(y+z,t)F(y,z)F^{-1}(z,t)\\
&\quad\times F^{-1}(y,z+t)F^{-1}(x,y+z+t) - F(x+y+z,t)F(x+y,z)\\
&\quad\times F^{-1}(z,t)F^{-1}(x+y,z+t)F(x+y,z+t)F(x,y)\\
&\quad\times F^{-1}(y,z+t)F^{-1}(x,y+z+t) \stackrel{n}{=} 0. \qquad \square
\end{aligned}$$

Proof of Proposition 1. Since we have $A(F,\alpha) \stackrel{n-1}{=} 0$, we have $(f(F))_{n-1} = (\alpha)_{n-1} \in (\otimes^3\mathcal{UG})^{\mathrm{inv}}[[h]]$ and, since $(F)_0 = 1\otimes 1$, then $[f(F)(x,y,z), F(x+y+z,t)] \stackrel{n}{=} 0$ and $[f(F)(y,z,t), F(x,y+z+t)] \stackrel{n}{=} 0$. Hence $P(f(F)) \stackrel{n}{=} 0$. But

$$[P(\alpha)]_n = d\alpha_n + p_n(\alpha),$$

where $p_n(\alpha)$ depends only on α up to order $n-1$.

Hence $da_n(F,\alpha) = p_n((\alpha)_{n-1})$ and $[P(\alpha)]_n = d\alpha_n + p_n(\alpha) = d(\alpha_n + a_n(F,\alpha))$. $\square$

3. Step by step deformation

We shall use here the Neroslavsky–Vlassov method to construct a star product in our situation. This is based on Hochschild cohomology and its splitting relative to a symmetry operator. Let us quite recall some classical results.

Theorem 1 (Vey [8]). *For any $z \in \otimes^k(\mathcal{UG})_0$ such that $dz = 0$, there exist $v \in \otimes^{k-1}(\mathcal{UG})_0$ and a unique $w \in \wedge^k\mathcal{G}$ such that $z = dv + w$. Furthermore w is the completely antisymmetric part of z ($w = AS(z)$).*

Remark. We can split the Hochschild cohomology with the following involution:

$$\sigma : \otimes^k(\mathcal{UG})_0 \to \otimes^k(\mathcal{UG})_0 : u_1\otimes\cdots\otimes u_k \to (-1)^{k(k+1)/2} u_k \otimes\cdots\otimes u_1$$

since this involution commutes with the Hochschild coboundary operator ($d\circ\sigma = \sigma\circ d$).

Theorem 2.

1. *Let $z \in \otimes^k(\mathcal{UG})_0$ such that $dz = 0$ and $\sigma(z) = (-1)^k z$ then there exist $v \in \otimes^{k-1}(\mathcal{UG})_0$ such that $\sigma(v) = (-1)^k v$ and $w \in \wedge^k\mathcal{G}$ such that $z = dv + w$. Furthermore w is the completely antisymmetric part of z ($w = AS(z)$).*

2. *Let* $z \in \otimes^k(\mathcal{U}\mathcal{G})_0$ *such that* $dz = 0$ *and* $\sigma(z) = (-1)^{k+1}z$ *then there exists* $v \in \otimes^{k-1}(\mathcal{U}\mathcal{G})_0$ *such that* $\sigma(v) = (-1)^{k+1}v$ *and* $z = dv$.

The first step of the Neroslavsky–Vlassov method to find a star product is to impose symmetry conditions on the bidifferential operators which appear in the star product such that, at each step, the cohomological problem is in one of the two cohomological spaces for this decomposition relative to σ. We shall try to do the same thing here.

Definition. Let A be an algebra. Define s to be the operator on $A[[h]]$ which maps $\sum_{i\geq 0} h^i a_i$ to $s(\sum_{i\geq 0} h^i a_i) := \sum_{i\geq 0}(-1)^i h^i a_i$.

From now on, we look for a solution (F, α) of the associativity equation ($A(F, \alpha) = 0$) such that $s \circ \sigma(F) + F = 0$. This condition implies that the odd order cochains are antisymmetric and that the even order ones are symmetric.

Assume that $F = 1\otimes 1 + \sum_{i=1}^{n-1} h^i F_i$ is a solution of the associativity equation to $(n-1)$th order ($A(F, \alpha) \overset{n-1}{=} 0$) such that $s \circ \sigma(F) + F \overset{n-1}{=} 0$. Then a necessary condition for the existence of a prolongation $F' = 1 \otimes 1 + \sum_{i=1}^{n} h^i F_i$ of this solution at order n such that $s \circ \sigma(F') + F' \overset{n}{=} 0$ is $\sigma(\alpha_n + a_n(F, \alpha)) + (-1)^n(\alpha_n + a_n(F, \alpha)) = 0$.

Proposition 2. *Let* $\alpha = 1 \otimes 1 \otimes 1 + \sum_{i\geq 1} h^i \alpha_i(x, y, z)$ *and* $F = 1 \otimes 1 + \sum_{i=1}^{n-1} h^i F_i$ *be such that* $s \circ \sigma(F) + F \overset{n-1}{=} 0$ *and* $A(F, \alpha) \overset{n-1}{=} 0$. *Then* $\sigma(\alpha_n + a_n(F, \alpha)) + (-1)^n(\alpha_n + a_n(F, \alpha)) = 0$ *if and only if* $T(\alpha) \overset{n}{=} 1$ *where*

$$T(\alpha) := \alpha \cdot s \circ \sigma(\alpha). \tag{5}$$

Lemma 3. *Let* $F = 1\otimes 1 + \sum_{i\geq 1}^{n-1} h^i F_i$ *be such that* $s\circ\sigma(F) + F \overset{n}{=} 0$; *then* $T(f(F)) \overset{n}{=} 1$.

Proof.

$$\begin{aligned} f(F) \cdot s \circ \sigma(f(F)) &\overset{n}{=} F(x+y, z)F(x, y)F^{-1}(y, z)F^{-1}(x, y+z) \\ &\quad \times (sF)(z+y, x)(sF)(z, y)(sF^{-1})(y, x)(sF^{-1})(z, y+x) \\ &\overset{n}{=} F(x+y, z)F(x, y)F^{-1}(y, z)F^{-1}(x, y+z)F(x, z+y) \\ &\quad \times F(y, z)F^{-1}(x, y)F^{-1}(y+x, z) \overset{n}{=} 1. \qquad \square \end{aligned}$$

Proof of Proposition 2. Since $s \circ \sigma(F) + F \overset{n}{=} 0$ then $T(f(F)) \overset{n}{=} 0$. But

$$[T(\alpha)]_n = \alpha_n + (-1)^n\sigma(\alpha_n) + t_n(\alpha),$$

where $t_n(\alpha)$ depends only on α until order $n-1$. Hence $a_n(F, \alpha) + (-1)^n\sigma(a_n(F, \alpha)) = t_n((\alpha)_{n-1})$ and

$$\begin{aligned} [T(\alpha)]_n &= \alpha_n + (-1)^n\sigma(\alpha_n) + t_n(\alpha) \\ &= (\alpha_n + a_n(F, \alpha)) + (-1)^n\sigma(\alpha_n + a_n(F, \alpha)) \qquad \square \end{aligned}$$

Theorem 3 (Drinfeld [4]). *Let α_2 be any element of $(\wedge^3\mathcal{G})^{\mathrm{inv}}$. Then there exists $\alpha = 1\otimes 1\otimes 1 + h^2\alpha_2 + \sum_{i\geq 2} h^{2i}\alpha_{2i}$, where $\alpha_i \in (\otimes^3(\mathcal{UG})_0)^{\mathrm{inv}}$, such that $P(\alpha)=0$ and $T(\alpha)=1$.*

We now have everything we want to try to use Neroslavsky–Vlassov's method.

At the first order the problem is to prove that there exist $\alpha_1 \in (\otimes^3(\mathcal{UG})_0)^{\mathrm{inv}}$ and $F_1 \in \otimes^2(\mathcal{UG})_0$ such that $\sigma(F_1)-F_1=0$, $F_1(x,y)-F_1(y,x)=r$ and $dF_1+\alpha_1=0$. $F_1 := \frac{1}{2}r$ and $\alpha_1 := 0$ are solutions.

At the second order the problem is to prove that there exist $\alpha_2 \in (\otimes^3(\mathcal{UG})_0)^{\mathrm{inv}}$ and $F_2 \in \otimes^2(\mathcal{UG})_0$ such that $\sigma(F_2)+F_2=0$ and $dF_2+\alpha_2+a_2(1\otimes 1+hF_1, 1\otimes 1\otimes 1)=0$.

Since $P(1\otimes 1\otimes 1)=0$ and $T(1\otimes 1\otimes 1)=1$ we have $da_2(1\otimes 1+hF_1, 1\otimes 1\otimes 1)=0$ and $\sigma(a_2(1\otimes 1+hF_1, 1\otimes 1\otimes 1))+a_2(1\otimes 1+hF_1, 1\otimes 1\otimes 1)=0$.

Then there exists a good F_2 if and only if $AS(\alpha_2+a_2(1\otimes 1+hF_1, 1\otimes 1\otimes 1))=0$ but $AS(a_2(1\otimes 1+hF_1, 1\otimes 1\otimes 1)) = -\frac{1}{4}[F_1,F_1] = -\frac{1}{16}[r,r] \in (\wedge^3\mathcal{G})^{\mathrm{inv}}$. Then, if we take $\alpha_2 = \frac{1}{16}[r,r]$, there exists a good F_2 and the problem is solved at the order 2.

Now let $\alpha = 1\otimes 1\otimes 1 + h^2\frac{1}{16}[r,r] + \sum_{i\geq 2} h^{2i}\alpha_{2i}$ such that $P(\alpha)=0$ and $T(\alpha)=1$ (we know that such an α exists using Theorem 3).

Let $F = 1\otimes 1 + \sum_{i=1}^{2n} h^i F_i$ be such that $s\circ\sigma(F)+F=0$ and $A(F,\alpha) \overset{2n}{=} 0$. Using Theorem 2, we can find $F' = 1\otimes 1 + \sum_{i=1}^{2n+1} h^i F_i$ an extension of F such that $s\circ\sigma(F')+F'=0$ and $A(F',\alpha) \overset{2n+1}{=} 0$.

Let $F = 1\otimes 1 + \sum_{i=1}^{2n+1} h^i F_i$ be such that $s\circ\sigma(F)+F=0$ and $A(F,\alpha) \overset{2n+1}{=} 0$. Using Theorem 3, we know that the only obstruction to the existence of an extension of F is $AS(\alpha_{2n+2}+a_{2n+2}(F,\alpha))$.

But if F is such that $s\circ\sigma(F)+F=0$ and $A(F,\alpha) \overset{2n+1}{=} 0$ then both equations are still true for $F' := F + h^{2n+1}w$ where $w\in\wedge^2\mathcal{G}$ and the only obstruction to the existence of an extension of F' is $AS(\alpha_{2n+2}+a_{2n+2}(F,\alpha)) + 3[F_1,w]$.

In particular we have the following theorem.

Theorem 4. *Let $(G,\{\,\}_r)$ be an exact Poisson-Lie group. Suppose that the linear application $\wedge^2\mathcal{G}\to\wedge^3\mathcal{G} : w \to [r,w]$ is onto then there exist $\alpha = 1\otimes 1\otimes 1 + \sum_{i\geq 1} h^i\alpha_i$, where $\alpha_i \in (\otimes^3(\mathcal{UG})_0)^{\mathrm{inv}}$, and $F = 1\otimes 1 + \frac{1}{2}hr + \sum_{i\geq 2} h^1 F_i$ such that $s\circ\sigma(F)+F=0$ and $A(F,\alpha)=0$.*

4. Hopf ⋆ deformation of our examples

Now, we shall use the result of Section 3 to prove that for each non-Abelian Lie group G, there exists a nontrivial deformation $(C^\infty(G), \star_h, \Delta)$ of $(C^\infty(G), \cdot, \Delta)$.

In [3], we construct for each non-Abelian Lie group G an exact nontrivial Lie-Poisson structure $(G,\{\,\}_r)$. Let $(G,\{\,\}_r)$ be one of these exact Poisson-Lie group. We want to prove that, for each of them, there exist $\alpha = 1\otimes 1\otimes 1 + \sum_{i\geq 1} h^i\alpha_i$, where $\alpha_i \in (\otimes^3(\mathcal{UG})_0)^{\mathrm{inv}}$, and $F = 1\otimes 1 + \frac{1}{2}hr + \sum_{i\geq 2} h^i F_i$, where $F_i \in \otimes^2(\mathcal{UG})_0$, such that $A(F,\alpha)=0$.

If $[r, r] = 0$, Drinfeld has proved that there exists $F = 1 \otimes 1 + \frac{1}{2}hr + \sum_{i\geq 2} h^i F_i$, where $F_i \in \otimes^2(\mathcal{UG})_0$, such that $A(F, 1 \otimes 1 \otimes 1) = 0$, so the problem of Hopf star deformation is solved by this Drinfeld's solution.

If $[r, r] \neq 0$, the Lie-Poisson structure is one of the following:

1. $\mathcal{G} = su(2) = \rangle X, Y, Z\langle$ and the element r is $X \wedge Y$.
2. $\mathcal{G}$ is solvable, $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}] = Z(\mathcal{G})$ is one-dimensional and spanned by an element E, and the element r is $X \wedge Y$ where $[X, Y] = E$.
3. $\mathcal{G}$ is solvable, $Z(\mathcal{G}) = 0$, $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}]$ is Abelian, $\dim(\mathcal{G}^1) = 2$ and there exists a bilinear form β such that $ad\, x \circ ad\, y(z) = \beta(x, y)z$ for all $x, y \in \mathcal{G}$ and for all $z \in \mathcal{G}^1$. The element r is $X^1 \wedge X$, where $X^1 \in \mathcal{G}^1$ and $X \in \mathcal{G} \setminus \mathcal{G}^1$, such that $\beta(X, X) \neq 0$.

Remark. Let $(G, \{\ \}_r)$ be one of those described in the three previous cases. Then there exists a three-dimensional ideal $\mathcal{H}$ of $\mathcal{G}$ such that $r \in \wedge^2\mathcal{H} \subset \wedge^2\mathcal{G}$ and $(Int(\mathcal{G}))|_{\mathcal{H}} = Int(\mathcal{H})$ where $(Int(\mathcal{G}))|_{\mathcal{H}} := \{ad(x)|_{\mathcal{H}} \mid x \in \mathcal{G}\}$ and $Int(\mathcal{H}) := \{ad(x)|_{\mathcal{H}} \mid x \in \mathcal{H}\}$.

Let $(G, \{\ \}_r)$ be one of these Lie-Poisson structures. Let $\mathcal{G}$ be its Lie algebra and $\mathcal{H}$ be the minimal ideal which supports r. Let H be the simply connected Lie group of Lie algebra $\mathcal{H}$. Then $(H, \{\ \}_r)$ is a Lie-Poisson structure. Since $\dim(\mathcal{H}) = 3$ and $[r, r] \neq 0$ the linear map $\wedge^2\mathcal{H} \to \wedge^3\mathcal{H} : w \to [r, w]$ is onto. Thus, using Theorem 3 there exist $\alpha = 1 \otimes 1 \otimes 1 + \sum_{i\geq 1} h^i\alpha_i$, where $\alpha_i \in (\otimes^3(\mathcal{UH})_0)^{\mathrm{inv}}$, and $F = 1 \otimes 1 + \frac{1}{2}hr + \sum_{i\geq 2} h^i F_i$, where $F_i \in \otimes^2(\mathcal{UH})_0$, such that $A(F, \alpha) = 0$. Since $(Int(\mathcal{G}))|_{\mathcal{H}} = Int(\mathcal{H})$ we have $(\otimes^3(\mathcal{UH})_0)^{\mathrm{inv}} \subset (\otimes^3(\mathcal{UG})_0)^{\mathrm{inv}}$ then $\alpha_i \in (\otimes^3(\mathcal{UG})_0)^{\mathrm{inv}}$ and (F, α) is a solution of the problem for the Poisson-Lie group $(G, \{\ \}_r)$.

Hence, we have proved:

Theorem 5. *Let G be a non-Abelian Lie group. Then there exists on G a nontrivial exact Lie-Poisson structure $(G, \{\ \}_r)$ and a Hopf$\star$ deformation $(C^\infty(G), \star_h, \Delta)$ of this Lie Poisson structure.*

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